

Density Distribution function

The π -mode operation mode of the cavity corresponds to a configuration such that the E-fields in the two cells have opposite directions at a given time. An electron in the rf gun experiences the longitudinal electric field

$$E_z(z, t) = E_o \sin(\omega t - kz + \phi_o) \quad (1)$$

where E_o is the peak field, $k \equiv 2\pi/\lambda$ (where λ is the wavelength associated to the E-field in the resonant cavity) is the rf wave number, and ϕ_o is the initial injection phase angle of the electron with respect to the E-field. Let $\phi(z, t) = \omega t - kz + \phi_o$ be the phase between the electron and the electric field. The equation of motion for the electron is the coupled differential equations [1]

$$\frac{d\gamma}{dz} = ak(\sin(\phi) + \sin(\phi + 2kz)) \quad (2)$$

$$\frac{d\phi}{dz} = k\left(\frac{\gamma}{\sqrt{\gamma^2 - 1}} - 1\right) \quad (3)$$

In the first equation, the second term on the right-hand side represents the backward propagation traveling wave needed to establish the standing wave in the resonant cavity. The parameter $a \equiv eE/2m_e c^2 k$ is the strength of the accelerating field. The Hamiltonian for the system is

$$H(\gamma, \phi) = -ak(\cos(\phi) + \cos(\phi + 2kz)) + k(\gamma - \sqrt{\gamma^2 - 1}) \quad (4)$$

I will try to derive the free energy equation of the electron from the probability density distribution function. If we consider the case where energy is high ($\gamma \gg 1$) and that there is no backward propagation, then the Hamiltonian of the gun system will be simplified to:

$$H(\phi) = H(x) = -ak \cos(x) \quad (5)$$

The evolution of the system can be described by a time dependent probability density distribution function $P(x, t)$ of the form:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left(\frac{\partial H(x)}{\partial x} P \right) + \frac{\partial^2}{\partial x^2} (QP) \quad (6)$$

Where Q is the fluctuation strength. This kind of partial differential equation is what is commonly referred to as the Fokker-Planck Equation and is mainly used in describing a system in a non statistical equilibrium case [2]. The nonsteady solution for the probability density distribution function $P(x, t)$ is:

$$P(x, t) = \frac{1}{\sqrt{4\pi Qt}} \exp\left(-\frac{(x + ak \sin(x)t)^2}{4Qt}\right) \quad (7)$$

The stationary solution is :

$$P_{stat} = \frac{me^{m\theta}}{e^m - 1} \quad (8)$$

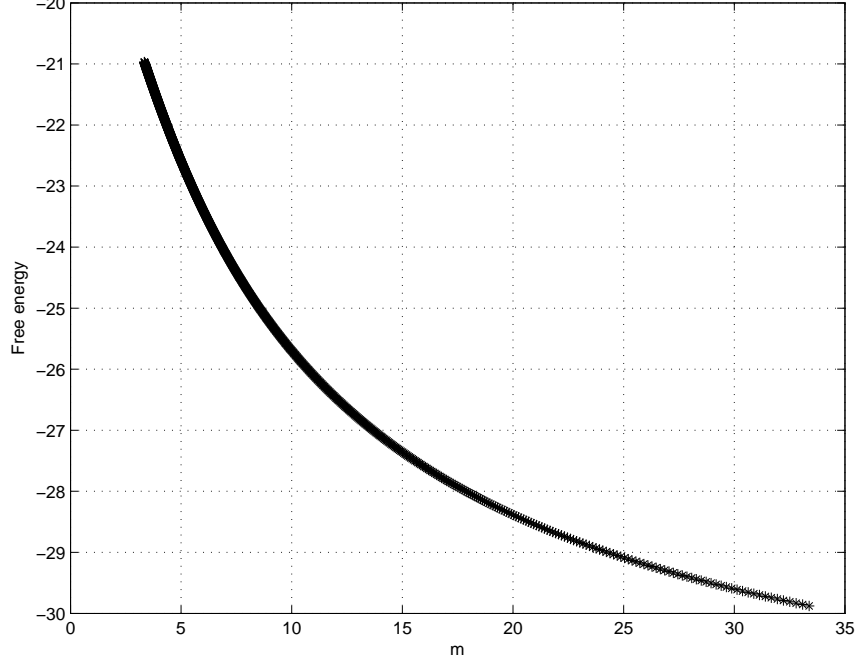


FIG. 1: Free energy F as a function of m the inverse of the fluctuation strength.

Where we introduce $m \equiv ak/Q$ and $\theta = \cos(x)$. The first order moments of the Hamiltonian can be calculated as:

$$\langle \theta \rangle = \int \theta P_{stat} d\theta = \frac{1 + e^m(m-1)}{m(e^m - 1)} \quad (9)$$

In order to define the *Free Energy* concept, Frank [3] suggested that the free energy F due to the entropy of the system is:

$$F = QI(P) \quad (10)$$

Where $I(P)$ is the Lyaponov functional for the stationary probability density distribution function P_{stat} and is given by:

$$I(p) = \int P_{stat} \ln\left(\frac{P_{stat}}{q}\right) d\theta \quad (11)$$

Where $q \equiv \exp(-\frac{H}{Q}) = \exp(m\theta)$, after some calculation, the free energy F for the system is:

$$F = \frac{ak}{m} \ln\left(\frac{m}{e^m - 1}\right) \quad (12)$$

Figure 1 is a plot for the free energy as a function of m the inverse of the fluctuation strength Q . We clearly can see that as the fluctuation increases (m gets smaller) in the system due to noise coming from a jitter or any other source, the free energy is higher. The energy usually dissipated as heat to the surrounding.

I. HALO FORMATION FOR GLUCKSTERN MODEL

The Guckstern model described a halo formation in high beam linacs. Halos are produced at transition locations, for example a defect or discontinuity in frequency[4]. we consider the case whre the space charge forces are linear and the charege distribution is uniform this is known as the K-V (Kapchinsky-Vladinisky) distribution. The simplified equation of motion for K-V (Kapchinsky-Vladinisky) circular beam in the x direction with a noise $\xi(z)$ added to the external force is:

$$x'' + (q^2 + 2\beta \cos(pz))x = (\alpha + \xi(z))x^3 \quad (13)$$

where $\alpha = \frac{-eI}{2\pi\epsilon_0 v^3 a^4}$ and $\beta = \frac{-e\epsilon I}{2\pi\epsilon_0 v^3 a^2}$. Here a is the K-V core radius, q is the wavenumber of oscillation, p is the core wavenumber, I,e,m,v are the charge charge, mass, and velocity of the particle, ϵ_0 is the permittivity of free space and ϵ is the first order approximation for the oscillation amplitude. An example of external force noise is a jitter in the machine. Using the phase amplitude method to find the Langevan equations, we have:

$$x = A \sin(qz + \phi) \quad (14)$$

Taking the first derivative with respect z :

$$x' = A \cos(qz + \phi) \quad (15)$$

Since A and ϕ are slowly variables then taking the first derivative with respect z :

$$x' = A' \sin(qz + \phi) + Aq \cos(qz + \phi) + A\phi' \cos(qz + \phi) \quad (16)$$

Compare the above two equations we get:

$$A' \sin(qz + \phi) + A\phi' \cos(qz + \phi) = 0 \quad (17)$$

from equation (3) we get:

$$x'' = A'q \cos(qz + \phi) - Aq^2 \sin(qz + \phi) - Aq\phi' \sin(qz + \phi) \quad (18)$$

plug the above equation in 13 and solve for A' and ϕ' we get:

$$qA' = \alpha A^3 \sin(qz + \phi) \cos(qz + \phi) - A\beta \sin(2(qz + \phi)) \cos(pz) + \overline{A^3 \xi(z) \sin^3(qz + \phi) \cos(qz + \phi)} \quad (19)$$

$$q\phi' = -\alpha A^2 \sin^4(qz + \phi) + 2\beta \sin^2(qz + \phi) \cos(pz) - \overline{A^2 \xi(z) \sin^4(qz + \phi)} \quad (20)$$

Now average over all rapid oscillations except the $(2q - p)$ terms, then:

$$A' = -\frac{A\beta}{q} \sin(\psi) + \overline{\xi(z) \sin^3(qz + \phi) \cos(qz + \phi)} \quad (21)$$

$$\phi' = -\frac{3\alpha A^2}{8q} - \frac{\beta}{2q} \cos(\psi) - \overline{\xi(z) \sin^4(qz + \phi)} \quad (22)$$

where $\psi = (2q - p) + 2\phi$ and the bar represent time averaging where we can write:

$$\overline{\xi(z) \sin^3(qz + \phi) \cos(qz + \phi)} = \langle \xi(z) \sin^3(qz + \phi) \cos(qz + \phi) \rangle + \zeta_1(z) \quad (23)$$

$$\overline{\xi(z) \sin^4(qz + \phi)} = \langle \xi(z) \sin^4(qz + \phi) \rangle + \zeta_2(z) \quad (24)$$

where ζ_1 and ζ_2 are randomly white noise whose intensities are k_1 and k_2 respectively, The $\langle \rangle$ are averaging over z . For simplicity in calculations, let:

$$a \equiv \overline{\langle \xi(z) \sin^3(qz + \phi) \cos(qz + \phi) \rangle} \quad (25)$$

$$b \equiv \overline{\langle \xi(z) \sin^4(qz + \phi) \rangle} \quad (26)$$

The Langavan equations for A and ψ are:

$$A' = \frac{-A\beta}{q} \sin(\psi) + \frac{a}{q} A^3 + \frac{A^3}{q} \zeta_1(z) \quad (27)$$

$$\psi' = (2q - p) - \frac{3\alpha A^2}{4q} - \frac{\beta}{q} \cos(\psi) - 2A^2 b - 2A^2 \zeta_2(z) \quad (28)$$

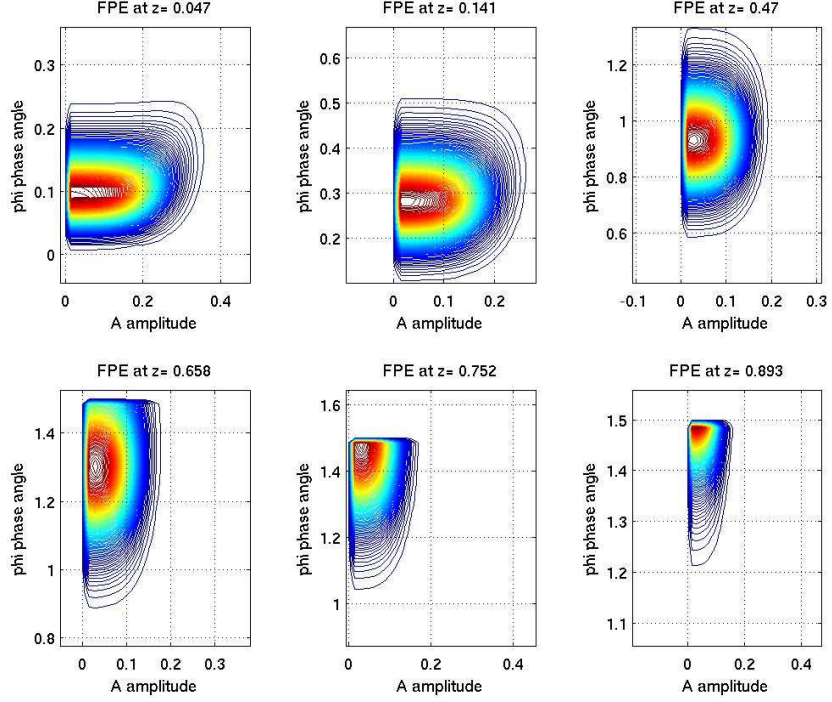


FIG. 2: Contour plot for the halo formation amplitude phase evolution with time.

The Fokker Planck Equation for the probability distribution function $u = u(A, \psi)$ associated with equations 27 and 28 is:

$$\begin{aligned} \frac{\partial u}{\partial t} = & -\frac{\partial}{\partial A} \left(\left(-\frac{A\beta}{q} \sin(\psi) + \frac{aA^3}{q} \right) u \right) + \frac{\partial^2}{\partial A^2} \left(\frac{A^6 k_1}{2q^2} u \right) \\ & -\frac{\partial}{\partial \psi} \left(\left((2q - p) - \frac{3\alpha A^2}{4q} - \frac{\beta}{q} \cos(\psi) - 2bA^2 \right) u \right) + \frac{\partial^2}{\partial \psi^2} (2k_2 A^4 u) \end{aligned} \quad (29)$$

Equation 29 was solved by Finite Difference Method and the routine for solving it was written in MATLAB. Figure 2 is a contour plot for the halo formation amplitude phase evolution with time. The parameters used in calculations are: $k_1 = 0.55, k_2 = 0.14, \alpha = -0.55, \beta = -0.08, a = 0.5, b = 0.2, p = 2.0$, and $q = 1.1$. Initial conditions and boundaries for the problem are: $u(A, 0) = 1$ for $A \leq 1$ else $u(A, 0) = 0$, and $u(0, \psi) = 0$ everywhere.

We also studied the case where the noise is added in the cosine term, hence the beam that we started with is noisy. An example could be the beam when it started at the photocathode it had some perturbations or not well aligned at the center of the beam longitudinal axis z . The equation of motion is:

$$x'' + (q^2 + 2\beta \cos((p + \xi(z))z))x = \alpha x^3 \quad (30)$$

Using the recipe described above (phase amplitude method), we can write the Langvian equations for 30:

$$A' = \frac{-A\beta}{q} \sin(\psi) + \frac{mA\beta}{q} + \frac{A\beta}{q} \zeta_1(z) \quad (31)$$

$$\psi' = (2q - p) - \frac{3\alpha A^2}{4q} - \frac{\beta}{q} \cos(\psi) - \frac{4\beta n}{q} - \frac{4\beta}{q} \zeta_2(z) \quad (32)$$

Below are the results for the Gluckstern model for:

- Case where no noise is considered, black dashed line.
- Case where noise is added to the external force, red solid line.
- Case where noise is added to the cosine term, blue solid line.

Figure 3 is a parametric plot for A and ψ . Figure 4 is a plot for the phase ψ versus time, note that the red line and the black dashed line are coincident. Figure 5 is a plot for Amplitude A versus time. We can clearly see that the amplitude of the beam is bigger when the core of the beam is noisy. The increase in amplitude of the beam may allow the beam to touch the beam pipe. This is dangerous since the process would trigger radiation that make it difficult to work around accelerators.

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- [1] K.-J. Kim, *Nucl. Instrum. Methods Phys. Res. Sec. A* **275**, 201 (1989).
[2] H. Risken. *The Fokker Planck Equation*, Springer, (1989)
[3] T. D. Frank, *H-Theorem for the Fokker Planck Equations with drifts depending on process mean values*, Elsevier Preprint, (2000)
[4] L. Robert Gluckstern, "Analytical Model for Halo Formation in High Current Ion Linacs", Physics Review Special Topics - Accelerators and Beams, Volume 7. 104202, (2004)

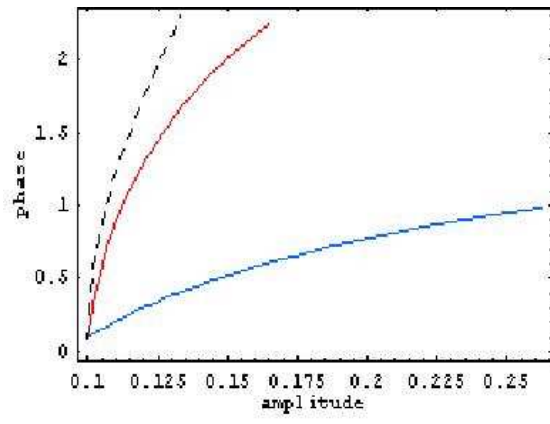


FIG. 3: Parametric plot for A and ψ

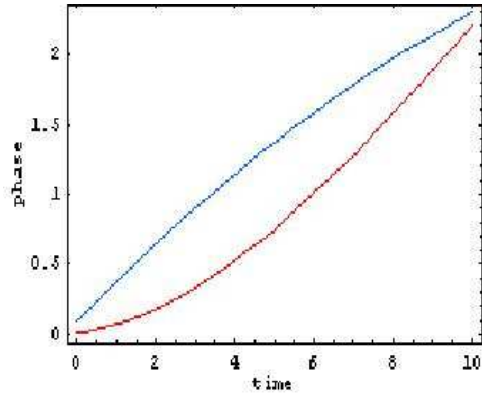


FIG. 4: Phase plot for ψ Vs time

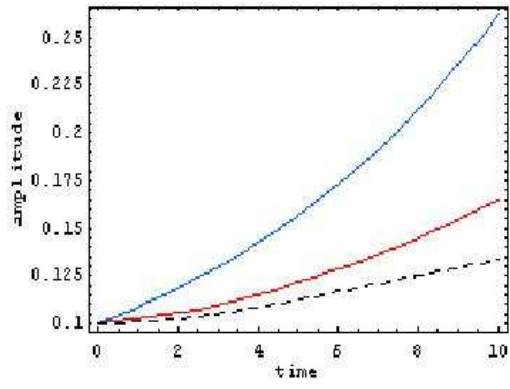


FIG. 5: Amplitude plot for A Vs time